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A REVIEW ON THE CHROMATIC NUMBER OF CARTESIAN PRODUCTS AND APPLICATIONS

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ABSTRACT

Given any graph G, its square graph G2 has the same vertex set V (G), with two vertices adjacent in G2 whenever they are at distance 1 or 2 in G. A set S ⊆ V (G) is a 2-distance independentset of a graph G if the distance between every two vertices of Sis greater than 2. The 2-distance independence number α2(G) of G is the maximum cardinality over all 2-distance independent sets in G. In this paper, we establish the 2-distance independence number and 2-distance chromatic number for C3>C6>Cm, Cn>P3>P3 and C4>C7>Cn where m ≡ 0 (mod 3)and n, m ⩾ 3.

Introduction

Let G = (V, E) be a finite and simple graph. For any graph G, we denote the vertex-set and the edge-set of G by V (G) and E(G), respectively. A proper vertex k-coloring of a graph G isa mapping c : V (G) → {1, . . . , k}, with the property that c(u) ≠ c(v) whenever uv ∈ E(G). The smallest k for which there exists a k-coloring of G, called the chromatic number of G, is denotedby χ(G), see [1, 7] for more details. The square of a graph G, denoted by G2 , is a graph with V (G) = V (G2), in which two vertices are adjacent if their distance in G is at most two. A 2- distance coloring of G is a vertex coloring of G such that any two distinct vertices at distance less than or equal to 2 are assigned different colors. The 2-distance chromatic number of a graph G is the minimum number of colors necessary to have a 2-distance coloring of G, which is denoted by χ2(G). Hence χ2(G) is equal to χ(G2). The 2-distance coloring of graphs was introduced by Wegner in [16]. The problem of determining thechromatic number of the square of particular graphs has attracted a lot of attention, with a particular focus on the square of planar graphs (see, e.g., [4, 5, 8, 10, 15]). The Cartesian product of graphs G1, G2, . . . , Gk is the graph G1,G2, · · ,Gk = k i=1Gi with vertex set {(x1, x2, . . . , xk)|xi ∈ V (Gi)} and for which two vertices (x1, x2, . . . , xk) and (y1, y2, . . . , yk) are adjacent whenever xiyi ∈ E(Gi) for exactly one index 1⩾ i ⩾ k and xj = yj for each index1 ⩾ j ⩾ k that i j. The subgraph of G>H induced by {u}×V (H) is isomorphic to H. It is called an H-fiber and is denoted by Hu . A set S ⊆ V (G)

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is ak-distance independent set of a graph G if the distance betweenevery two vertices of S is greater than k. The k-distance independence number αk(G) of G is the maximum cardinality over all k-distance independent sets in G. For k = 1, we use αk(G) as α(G). There are many results for the chromatic number of the square of the Cartesian product of tree, paths, and cycles (see, e.g., [2, 3, 6, 9, 11, 13]). Shao et al. [12] established that the 2-distance chromatic number of G equals ⌈|V (G)| α(G2) ⌉ for G = Cm>Cn>Ck where k ⩾ 3 and (m, n) ∈{(3, 3),(3, 4),(3, 5),(4, 4)} or k, m, and n are all multiples of seven. Moreover, it is shown that the 2-distance chromatic number of the three-dimensional square lattice is equal to sevenand proved the following theorems.

Theorem 1.1 [12] If j, k, l ⩾ 1, then

α2(C7j>C7k>C7l) = 49jkl.

Theorem 1.2 [12] If j, k, l ⩾ 1, then

χ2(C7j>C7k>C7l) = 7.

In this paper, as an extension of Theorems 1.1 and 1.2, we establish the 2- distance independence number and 2-distance chromatic number for C3>C6>Cm, Cn>P3>P3 and C4>C7>Cn where m ≡ 0 (mod 3) and n, m ⩾ 3.

1. Main results

The aim of this section is to find lower and upper bounds and exact values for the spcial cases 2-distance chromatic number of the families G = {C3>C6>Cm, Cn>P3>P3, C4>C7>Cn where m ≡ 0 (mod 3) and n, m ⩾ 3.} The following two lemmas are essential for proving the main theorems. Let Gbe a graph and f be a proper 2-coloring of G. Since every colorclass under f is a 2-independent set, we have the following lemma,

Lemma 2.1 If G is a graph, then χ2(G) ⩾ ⌈ |V (G)| α(G2). Let H be a graph, m ⩾ 3 and f denote a proper t-coloring of (Cm>H) 2 . We denote by fi,p, 0 ⩾ i ⩾ m − p and 1 ⩾ p ⩾ m, the restriction of f to V (Hi), . . . , V (Hi+p−1). The following lemma is a natural generalization of [11,Lemma 1].

Lemma 2.2 Let m, n, p ⩾ 3, s ⩾ 1 and let f be a proper t- coloring of (Cm>H) 2 . If f0,p is a proper t-coloring of (Cp>H)2 , then χ((Cm+(s−1)p>H) 2) ⩾ t.

Proof

Let f ′ : V (Cm+(s−1)p>H) −→ {1, 2, ..., k} be a function and f

′ i the restriction of f ′ to V (H^i). We define the function f ′ by

$$f'_i = \begin{cases} f_i & i < m, \\ f_{(i-m)\,mod\,p} & i \geqslant m. \end{cases}$$

Consider first the vertex (j, m). In this case vertex (j, m) is adjacent to {(j− l, m − 1); l ∈ {0, 1, −1}} and (j, m − 2) in the subgraph induced by V (H0), ..., V (Hm−1), as illustrated in Figure 1. By definition f ′ i we have f ′ (j, m) = f(j, 0). Since f is a propert-coloring of (Cm>H) 2 and (j, 0) is adjacent to {(j − l, m − 1);l ∈ 0, 1, −1} and (j, m − 2) in (Cn>H) 2 , this case is settled. Similarly for any two adjacent vertices (x, y) and (x ′ , y′) {(j, m + 1),(j, m + sp),(j, m + sp + 1), s1} of V(Cm+(s−1)p>H) 2 , we have f ′ (x, y)f ′ (x ′ , y′) and can be proved analogously. Therefore the proof is completed.

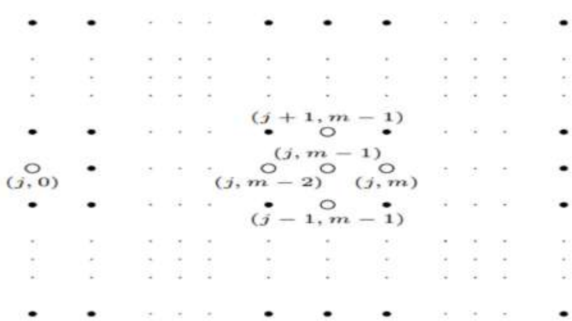

Figure 1. vertex-set of $(C_{m+(s-1)p}\Box H)^2$ for $s \geqslant$

Before presenting our main results we need to obtain the 2-distance independent number of families G. We first mention two lemmas that need for proof of next lemmas. Let H be a graph. If I is a d-distance independent set of Ck>H, then, for i = 0, . . . , k − 1, we set I i:= I ∩ V (H^i), that is, I i is the subset of I induced by the verticesof H^i .

One naturally asks whether an analogous statement holds for list coloring of graphs. Since the product of K1,2 and K1,4 contains the complete bipartite graph K2,4 and X_l(K2,4)= 3, the statement can hold with the maximum of the list chromatic numbers. Hence, one can at least ask whether the list chromatic number of G × H can be bounded by max{X_l(G), X_l(H)}+C for a constant C (or even for C=1).We show that even such a statement is false by constructing graphs G with X_l(G × G) = 2X_l(G) − 1. Another graph parameter closely related to the chromatic number and the list chromatic number is the coloring number. The coloring number col(G) of a graph G is the smallest integer d for which there exists an ordering v1,...,vn of the vertices of G such that each vertex vi has at most d − 1 neighbors among the vertices v1,...,vi−1. A graph G with col(G) = d is also called (d − 1)-degenerate. Clearly, (G)col(G) and l(G)col(G). Our main result is the following upper bound on the list chromatic number of the Cartesian product of two graphs G and H:

$$\chi_l(G \times H) \leqslant \min\{\chi_l(G) + \mathrm{col}(H), \mathrm{col}(G) + \chi_l(H)\} - 1.$$

The bound can be generalized to products of more graphs (see Corollary 2). In Section 3, we show that this bound cannot be improved. In particular, for every pair of positive integers k and , there exist a graph

G with l(G) = col(G) = k and a graph H with $X_l(H)$ = col(H) = such that $X_l(G \times H) = k + - 1$.

Upper bound

We start with establishing our upper bound on the list chromatic number of the Cartesian product of graphs.

Theorem 1

Let G and H be two graphs. The list chromatic number l(G × H) of the Cartesian product G × H can be bounded as follows:

$$\chi_l(G \times H) \leqslant \min\{\chi_l(G) + \mathrm{col}(H), \mathrm{col}(G) + \chi_l(H)\} - 1.$$

Proof

By symmetry, it is enough to prove that $\chi_l(G \times H) \leqslant \chi_l(G) + \mathrm{col}(H) - 1$. Let $v_1, \ldots, v_n$ be the vertices of H ordered in such a way that each vertex vi has at most col(H) − 1 neighbors among the vertices v1,...,vi−1. Let Vi be the vertices of G × H that are contained in the copy of G corresponding to a vertex vi. Fix a list assignment L for G × H such that |L(v)| = l(G) + col(H) − 1 for every vertex v ∈ V (G × H). We construct a proper coloring c of G × H with c(v) ∈ L(v) for every v ∈ V (G × H). First, color the subgraph of G × H induced by V1. Since this subgraph is isomorphic to G and each vertex has a list of size at least l(G), such a coloring exists. Assume that we have already constructed a proper coloring c of the subgraph of G × H induced by V1 ∪···∪ Vi−1. We now extend the coloring c to the vertices of Vi. First, remove from the list L(v) of each vertex v of Vi the colors of its neighbors among the vertices contained in V1 ∪···∪Vi−1. Since the vertex v has at most col(H)−1 such neighbors (one neighbor for each neighbor of vi that precedes vi in the ordering of the vertices of H), the new list L (v) has size at least l(G). Since the list chromatic number of G is l(G), the copy of G induced in G × H by the vertices of Vican be colored from the new lists. In this way, the coloring is eventually extended to the entire graph G × H.

An immediate corollary of Theorem 1 is the following upper bound on the list chromatic number of the Cartesian product of several graphs:

Corollary 2. If G1,...,Gk are graphs, then the following holds:

$$\chi_l(G_1 \times \cdots \times G_k) \leqslant \chi_l(G_1) + \mathrm{col}(G_2) + \cdots + \mathrm{col}(G_k) - (k-1).$$

2. Lower bound

In this section, we show that there exists a graph G with col(G) = l(G) such that l(G × G) = col(G) + l(G) − 1. Let us start with the following lemma:

Lemma 3

Let G be a graph with n vertices. The list chromatic number of the product of G and Kk,t is l(G) + k where t = $(k + l(G) - 1)\, k^n$

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Proof

Let H be the Cartesian product of G and Kk,t . Fix a list assignment L that assigns each vertex of G a set of l(G) − 1 colors such that the vertices of G cannot be properly colored from their lists. Let L0 be the union of all the lists L(v) and let X be the smaller part of Kk,t and Y the larger one. Finally, let XH be the vertices of H that are contained in the copies of G corresponding to the vertices of X. Note that |XH | = kn.

We now construct a list assignment LH from which H cannot be colored. The lists LH (v, x), v ∈ V (G) and x ∈ X, i.e., the lists of the vertices of XH , are disjoint sets of k + l(G) − 1 that are distinct from the colors of L0. Next, associate with each of the (at most t = (k + l(G) − 1) kn) colorings c of the vertices of XH a vertex yc of Y. The list LH (v, yc) is the union of the list L(v) and the set of k colors assigned to the k neighbors of the vertex (v, yc) in XH . Observe that the size of the list LH (w) is k +l(G)−1 for every vertex w ∈ V (H). We show that H cannot be colored from the lists LH .

Assume that there exists a coloring cH of H such that cH (w) ∈ LH (w)for every w ∈ V (H). Let c be the restriction of cH to the vertices of XH . Observe now that cH (v, yc) ∈ L(v) for every vertex v: indeed, cH (v, yc) cannot be any of the k colors assigned to the neighbors of (v, yc) in X0. Since these k colors are precisely the k colors of LH (v, yc)\L0(v), it follows that cH (v, yc) ∈ L(v). Hence, the coloring cH restricted to the copy of G corresponding to the vertex yc in H is a proper coloring of G from the lists L. This contradicts the choice of the list assignment L.

Since H cannot be colored from the lists LH , l(H) >l(G)+k−1. Since the graph Kk,t is k-degenerate, its coloring number is k + 1 and l(H)l(G) + k by Theorem 1.

Hence, l(H) = l(G) + k.

H = K ,t with s = (k +) k (+) and t = (k +) (k +kk) .

4. Open problems We have initiated study of the list chromatic number of the Cartesian product of two graphs. Our original motivation was the question whether the list chromatic number l(G × H) of two graphs G and H could be bounded by max{l(G), l(H)} as in the case of usual colorings. We have shown that this does not hold for list colorings, in particular, l(G × G) = 2l(G) − 1 for the graph G constructed in Theorem 4. However, l(G × H) can be bounded by a function of l(G) and l(H): by the result of Alon [2,1], the coloring number of G does not exceed 2O(l(G)). Similarly, col(H)2O(l(H)). Hence, col(G × H) min{l(G) + 2O(l(H)), l(H) + 2O(l(G))}. However, we suspect that a much better upper bound can be established:

CONCLUSION

There exists a constant A such that the following holds for every pair of graphs G and H:

$$\chi_l(G \times H) \leqslant A(\chi_l(G) + \chi_l(H)).$$

Also note that if the (am, bm)-conjecture of Erd ″os et al. [3] is true, then l(G × H)l(G)l(H). Another problem is to bound the list chromatic number of G × H in terms of the maximum degrees of G and H. If G and H are complete graphs of orders a and b, then G × H is isomorphic to the line graph of the complete bipartite graph with parts of sizes a and b. Kahn [4] showed that the list chromatic number of the line-graph of a graph with maximum degree does not exceed + o(). In particular, l(Ka × Kb) = max{a, b} + o(a + b). This leads us to the following problem: Conjecture 7. Let G and H be two graphs with maximum degree at most . The list chromatic number of G × H does not exceed + o().

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