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Fixed Point Theorems for Modified Intuitionistic Fuzzy Metric Space

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Abstract

In this paper, we establish common fixed point theorems for mappings that are occasionally weakly compatible within the framework of a modified intuitionistic fuzzy metric space. As a result, our findings enhance and refine several existing common fixed point theorems reported in the literature on metric fixed point theory.

Keywords

Modified Intuitionistic Fuzzy Metric Space, Common Fixed Point, Fuzzy Metric Space, t −representable norms, Occasionally Weakly Compatible Mappings.

1 Introduction

The notion of fuzzy sets was first introduced by Zadeh [7], laying the foundation for extensive developments in fuzzy topology and analysis. Building on this, Park [5] formulated the concept now recognized as the **intuitionistic fuzzy metric space**, which incorporates the ideas of continuous norm and conorm. Over the years, this framework has found significant applications across various mathematical and scientific domains. Fixed point theory, in particular, stands out as a powerful and widely applicable branch of mathematics, playing a crucial role in fields such as chaos theory, game theory, and the study of differential equations. The intuitionistic fuzzy metric framework has also proven effective in modeling physical phenomena, especially where analyzing the interaction between two probability functions is essential—as highlighted by Gregori et al. [4]. A notable example is its application in explaining the double-slit experiment, serving as a foundational element of the infinity theory in high-energy physics, as discussed by El Naschie [1, 2, 3]. Gregori et al. [4] also observed that the topology derived from intuitionistic fuzzy metrics aligns with that of traditional fuzzy metric spaces. Building upon this, Saadati et al. [6] redefined the concept and introduced the **Modified Intuitionistic Fuzzy Metric Space** by incorporating the idea of continuous representability, further enriching the theoretical landscape. In this chapter, we present common fixed point theorems for occasionally weakly compatible mappings within the framework of modified intuitionistic fuzzy metric spaces. Our results not only generalize but also refine many existing theorems in the domain of metric fixed point theory.

2 Preliminaries

Definition 2.1: A binary operation $*: [0,1] \times [0,1] \to [0,1]$ is a continuous $t - norm$ if $*$ satisfies following conditions:

(i) $*$ is commutative and associative;
(ii) $*$ is continuous;
(iii) $a * 1 = a$ for all $a \in [0,1]$;
(iv) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, and $a, b, c, d \in [0,1]$.

Definition 2.2: A binary operation $\lozenge: [0,1] \times [0,1] \longrightarrow [0,1]$ is a continuous t −conorm if $\lozenge$ is satisfying the following conditions

(i) $\lozenge$ is commutative and associative;
(ii) $\lozenge$ is continuous;
(iii) $a \lozenge 0 = a$ for all $a \in [0,1]$;
(iv) $a \lozenge b \leq c \lozenge d$ whenever $a \leq c$ and $b \leq d$, for all $a, b, c, d \in [0,1]$.

Definition 2.3: Let $\mathcal{L} = (L^*, \leq_{L^*})$ be a complete lattice, and U a non empty set called a universe. An $\mathcal{L}$ −fuzzy set $\mathcal{A}$ on U is defined as amapping $\mathcal{A}: U \to L^*$. For each u in $U, \mathcal{A}(u)$ represents the degree (in L^*) to which u satisfies $\mathcal{A}$.

Lemma 2.1: Consider the set L^* and operation $\leq_{L^*}$ defined by :

$$L^* = \{(x_1, x_2) : (x_1, x_2) \in [0,1]^2 \text{ and } x_1 + x_2 \leq 1\},$$

$$(x_1, x_2) \leq_{L^*} (y_1, y_2) \Leftrightarrow x_1 \leq y_1$$

and $x_2 \geq y_2$, for every $(x_1, x_2), (y_1, y_2) \in L^*$.

then $(L^*, \leq_{L^*})$ is a complete lattice.

Definition 2.4: An intuitionistic fuzzy set $\mathcal{A}_{\zeta,\eta}$ in a universe U is an object $\mathcal{A}_{\zeta,\eta} = \{(\zeta_{\mathcal{A}}(u), \eta_{\mathcal{A}}(u)) | u \in U\}$, where, for all $u \in U$, $\zeta_{\mathcal{A}}(u) \in [0,1]$ and $\eta_{\mathcal{A}}(u) \in [0,1]$ are called the membership degree and the non-membership degree, respectively of u in $\mathcal{A}_{\zeta,\eta}$ and furthermore they satisfy $\zeta_{\mathcal{A}}(u) + \eta_{\mathcal{A}}(u) \leq 1$.

Definition 2.5: For every $z_\alpha = (x_\alpha, y_\alpha) \in L^*$ we define

$$\vee (z_\alpha) = \big(sup(x_\alpha), inf(y_\alpha)\big).$$

Since $z_\alpha \in L^*$ then $x_\alpha + y_\alpha \leq 1$

so $sup(x_\alpha) + inf(y_\alpha) \leq sup(x_\alpha + y_\alpha) \leq 1$,

that is $\vee (z_\alpha) \in L^*$. We denote its units by $0_{L^*} = (0,1)$ and $1_{L^*} = (1,0)$.

Classically, a triangular norm $* = T$ on $([0,1], \leq)$ is define as an increasing, commutative, associative mapping $T: [0,1]^2 \to [0,1]$ satisfying $T(1, x) = 1 * x = x$, for all $x \in [0,1]$. A triangular conorm $S = \lozenge$ is defined as an increasing, commutative, associative mapping $S: [0,1]^2 \to [0,1]$ satisfying $S(0, x) = 0 \lozenge x = x$, for all $x \in [0,1]$. Using the lattice $(L^*, \leq_{L^*})$ these definitions can be straightforwardly extended.

Definition 2.6: A triangular norm (t–norm) on L^* is a mapping

$\mathcal{T} : (L^*)^2 \to L^*$ satisfying the following conditions:

1) $(\forall x \in L^*)(\mathcal{T}(x, 1_{L^*}) = x)$, (boundary condition)
2) $(\forall (x,y) \in (L^*)^2)(\mathcal{T}(x,y) = \mathcal{T}(y,x))$, (commutativity)
3) $(\forall (x,y,z) \in (L^*)^3)\left(\mathcal{T}(x, \mathcal{T}(y,z)) = \mathcal{T}(\mathcal{T}(x,y),z)\right)$, (associativity),
4) $(\forall (x, x', y, y') \in (L^*)^4)(x \leq_{L^*} x')$ and $\left(y \leq_{L^*} y' \Rightarrow \mathcal{T}(x,y) \leq_{L^*} \mathcal{T}(x', y')\right)$. (monotonicity).

Definition 2.7: A continuous t–norm $\mathcal{T}$ on L^* is called continuous t–representable if and only if there exist a continuous t–norm $*$ and a continuous t–conorm $\lozenge$ on $[0,1]$ such that, for all $x = (x_1, x_2)$, $y = (y_1, y_2) \in L^*$,

$$\mathcal{T}(x,y) = (x_1 * y_1, x_2 \lozenge y_2).$$

Now define a sequence $\mathcal{T}^n$ recursively by $\mathcal{T}^1 = \mathcal{T}$ and

$$\mathcal{T}^n\left(x^{(1)}, \cdots, x^{(n+1)}\right) = \mathcal{T}\left(\mathcal{T}^{n-1}\left(x^{(1)}, \cdots, x^{(n)}\right), x^{(n+1)}\right)$$

For $n \geq 2$ and $x^{(i)} \in L^*$.

We say the continuous t–representable norm is natural and write $\mathcal{T}_n$ whenever

$\mathcal{T}_n(a,b) = \mathcal{T}_n(c,d)$ and $a \leq_{L^*} c$ implies $b \leq_{L^*} d$.

Definition 2.8: A negator on L^* is any decreasing mapping $\mathcal{N} : L^* \to L^*$ satisfying $\mathcal{N}(0_{L^*}) = 1_{L^*}$ and $\mathcal{N}(1_{L^*}) = 0_{L^*}$. If $\mathcal{N}(\mathcal{N}(x)) = x$, for all $x \in L^*$, then $\mathcal{N}$ is called an involutive negator. A negator on $[0,1]$ is a decreasing mapping $N: [0,1] \to [0,1]$ satisfying $N(0) = 1$ and $N(1) = 0$. N_S denotes the standard negator on $[0,1]$ defined as $N_S(x) = 1 - x$ for all $x \in [0,1]$.

Definition 2.9: Let M, N are fuzzy sets from $X^2 \times (0, +\infty)$ to $[0,1]$such that $M(x,y,t) + N(x,y,t) \leq 1$ for all $x, y \in X$ and $t > 0$. The $3-$tuple $\left(X, \mathcal{M}_{M,N}, \mathcal{T}\right)$ is said to be a Modified intuitionistic fuzzy metric space if X is an arbitrary (non-empty) set, $\mathcal{T}$ is a continuous $t-$representable norm and $\mathcal{M}_{M,N}$ is a mapping $X^2 \times (0, +\infty) \to L^*$ (an intuitionistic fuzzy set, see Definition 7.4) satisfying the following conditions for every $x, y \in X$ and $t, s > 0$:

(i) $\mathcal{M}_{M,N}(x,y,t) >_{L^*} 0_{L^*}$;
(ii) $\mathcal{M}_{M,N}(x,y,t) = 1_{L^*}$ if and only if $x = y$;
(iii) $\mathcal{M}_{M,N}(x,y,t) = \mathcal{M}_{M,N}(y,x,t)$;
(iv) $\mathcal{M}_{M,N}(x,y,t+s)$

$$\geq_{L^*} \mathcal{T}\left(\mathcal{M}_{M,N}(x,z,t), \mathcal{M}_{M,N}(z,y,s)\right);$$

(v) $\mathcal{M}_{M,N}(x,y,\cdot): (0,\infty) \to L^*$ is continuous.

In this case $\mathcal{M}_{M,N}$ is called a Modified intuitionistic fuzzy metric space. Here,

$\mathcal{M}_{M,N}(x,y,t) = (M(x,y,t), N(x,y,t))$.

In the sequel, we will call $\left(X, \mathcal{M}_{M,N}, \mathcal{T}\right)$ to be just a Modified intuitionistic fuzzy metric space.

Remark 2.1: In a Modified intuitionistic fuzzy metric space

$(X, \mathcal{M}_{M,N}, \mathcal{T}), \mathcal{M}(x, y, .)$ is non decreasing and $N(x, y, .)$ is non increasing for all $x, y \in X$. Hence $\mathcal{M}_{M,N}(x, y, t)$ is non decreasing with respect to t for all $x, y \in X$.

Example 2.1: Let (X, d) be a metric space. Define $\mathcal{T}(a, b) = \{a_1 b_1, \min(a_2 + b_2, 1)\}$ for all $a = (a_1, a_2)$ and $b = (b_1, b_2) \in L^*$ and let M and N be fuzzy sets on $X^2 \times (0, \infty)$ defined as follows:

$$\mathcal{M}_{M,N}(x, y, t) = \big(M(x, y, t), N(x, y, t)\big) = \left(\frac{ht^n}{ht^n + md(x, y)}, \frac{md(x, y)}{ht^n + md(x, y)}\right),$$

for all $h, m, n, t \in \mathbb{R}^+$. Then $(X, \mathcal{M}_{M,N}, \mathcal{T})$ is a modified intuitionistic fuzzy metric space.

Example 2.2: Let $X = N$. Define $\mathcal{T}(a, b) = \{(\max(0, a_1 + b_1 - 1), a_2 + b_2 - a_2 b_2\}$ for all $a = (a_1, a_2)$ and $b = (b_1, b_2) \in L^*$ and let M and N be fuzzy sets on $X^2 \times (0, \infty)$ defined as follows:

$$\mathcal{M}_{M,N}(x, y, t) = \big(M(x, y, t), N(x, y, t)\big) = \begin{cases} \left(\frac{x}{y}, \frac{y - x}{y}\right) & if\ x \le y \\ \left(\frac{y}{x}, \frac{x - y}{x}\right) & if\ y \le x \end{cases}$$

for all $x, y \epsilon X$ and $t > 0$. Then $(X, \mathcal{M}_{M,N}, \mathcal{T})$ is a Modified intuitionistic fuzzy metric space.

Example 2.3: Let $(X, \mathcal{M}_{M,N}, \mathcal{T})$ be a Modified intuitionistic fuzzy metric space. Where $X = [0, 2]$ and

$$\mathcal{M}_{M,N}(x, y, t) = \left(\frac{1}{t + d(x, y)}, \frac{d(x, y)}{t + d(x, y)}\right)$$

for all $t > 0$ and $x, y \in X$.

Denote $\mathcal{T}(a, b) = \{a_1 b_1, \min(a_2 + b_2, 1)\}$ for all $a = (a_1, a_2)$ and $b = (b_1, b_2) \in L^*$

Definition 2.10: Let $(X, \mathcal{M}_{M,N}, \mathcal{T})$ be a Modified intuitionistic fuzzy metric space and $\{x_n\}$ be a sequence in X.

(i) A sequence $\{x_n\}$ is said to be convergent to $x \in X$ in the Modified intuitionistic fuzzy metric space $(X, \mathcal{M}_{M,N}, \mathcal{T})$ and denoted by $x_n \xrightarrow{\mathcal{M}_{M,N}} x$ if $\mathcal{M}_{M,N}(x_n, x, t) \to 1_{L^*}$ as $n \to \infty$ for every $t > 0$.

(ii) A sequence $\{x_n\}$ in a Modified intuitionistic fuzzy metric space $(X, \mathcal{M}_{M,N}, \mathcal{T})$ is called a Cauchy sequence if for each $0 < \varepsilon < 1$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $\mathcal{M}_{M,N}(x_n, y_m, t) >_{L^*} (N_s(\varepsilon), \varepsilon)$, and for each $n, m \ge n_0$; here N_s is the standard negator.

(iii) A Modified intuitionistic fuzzy metric space is said to be complete if and only if every Cauchy sequence in this space is convergent. Henceforth, we assume that $\mathcal{T}$ is a continuous t –norm on the lattice $\mathcal{L}$ such that for every $\mu \in L^* \setminus \{0_{L^*}, 1_{L^*}\}$, there exists $\lambda \in L^* \setminus \{0_{L^*}, 1_{L^*}\}$ such that $\mathcal{T}^{n-1}(\mathcal{N}(\lambda), \cdots, \mathcal{N}(\lambda)) \ge_{L^*} \mathcal{N}(\mu)$.

Lemma 2.2: Let $\mathcal{M}_{M,N}$ be a Modified intuitionistic fuzzy metric. Then for any $t > 0$, $\mathcal{M}_{M,N}(x,y,t)$ is nondecreasing with respect to t in $(L^*, \leq_{L^*})$ for all $x, y \in X$.

Lemma 2.3 : Let $(X, \mathcal{M}_{M,N}, \mathcal{T})$ be a Modified intuitionistic fuzzy metric space and define $E_{\lambda,\mathcal{M}_{M,N}} : X^2 \to \mathbb{R}^+ \cup \{0\}$ by

$$E_{\lambda,\mathcal{M}_{M,N}}(x,y) = inf\{t > 0 : \mathcal{M}_{M,N}(x,y,t) >_{L^*} \mathcal{N}(\lambda)\}$$

for each $\lambda \in L^* \setminus \{0_{L^*}, 1_{L^*}\}$ and $x, y \in X$ here, $\mathcal{N}$ is an involutive negator. Then we have

(i) For any $\mu \in L^* \setminus \{0_{L^*}, 1_{L^*}\}$, there exists $\lambda \in L^* \setminus \{0_{L^*}, 1_{L^*}\}$ such that

$E_{\mu,\mathcal{M}_{M,N}}(x_1,x_n) \leq E_{\lambda,\mathcal{M}_{M,N}}(x_1,x_2) + E_{\lambda,\mathcal{M}_{M,N}}(x_2,x_3) + \cdots + E_{\lambda,\mathcal{M}_{M,N}}(x_{n-1},x_n)$
for any $x_1, x_2, x_3, \cdots x_n \in X$.

(ii) The sequence $\{x_n\}_{n\in\mathbb{N}}$ in X is convergent to x with respect to Modified intuitionistic fuzzy metric $\mathcal{M}_{M,N}$ if and only if $E_{\lambda,\mathcal{M}_{M,N}}(x_n,x) \to 0$. Also the sequence $\{x_n\}$ is a Cauchy sequence with respect to Modified intuitionistic fuzzy metric $\mathcal{M}_{M,N}$ if and only if it is a Cauchy sequence with $E_{\lambda,\mathcal{M}_{M,N}}$.

Proof. For (i), by the continuity of t −norms, for every $\mu \in L^* \setminus \{0_{L^*}, 1_{L^*}\}$, we can find a $\lambda \in L^* \setminus \{0_{L^*}, 1_{L^*}\}$ such that

$\mathcal{T}(\mathcal{N}(\lambda), \mathcal{N}(\lambda)) \geq_{L^*} \mathcal{N}(\mu)$.

By Definition 2.9 (iv), we have

$$\mathcal{M}_{M,N}(x, y, E_{\lambda,\mathcal{M}_{M,N}}(x_1,x_2) + E_{\lambda,\mathcal{M}_{M,N}}(x_2,x_3) + \cdots + E_{\lambda,\mathcal{M}_{M,N}}(x_{n-2},x_{n-1}) + E_{\lambda,\mathcal{M}_{M,N}}(x_{n-1},x_n) + n\delta)$$

$$\geq_{L^*} \mathcal{T}\Big(\mathcal{M}_{M,N}\big(x, z, E_{\lambda,\mathcal{M}_{M,N}}(x_1,x_2) + E_{\lambda,\mathcal{M}_{M,N}}(x_2,x_3) + \cdots + E_{\lambda,\mathcal{M}_{M,N}}(x_{n-3},x_{n-2}) + \frac{n\delta}{2}\big), \mathcal{M}_{M,N}\big(z, y, E_{\lambda,\mathcal{M}_{M,N}}(x_{n-2},x_{n-1}) + E_{\lambda,\mathcal{M}_{M,N}}(x_{n-1},x_n) + \frac{n\delta}{2}\big)\Big) \geq_{L^*} \mathcal{T}(\mathcal{N}(\lambda), \mathcal{N}(\lambda)) \geq_{L^*} \mathcal{N}(\mu)$$

for every $\delta > 0$, which implies that

$E_{\mu,\mathcal{M}_{M,N}}(x_1,x_n) \leq E_{\lambda,\mathcal{M}_{M,N}}(x_1,x_2) + E_{\lambda,\mathcal{M}_{M,N}}(x_2,x_3) + \cdots + E_{\lambda,\mathcal{M}_{M,N}}(x_{n-1},x_n) + n\delta$

Since $\delta > 0$ was arbitrary,

Now we have

$$E_{\mu,\mathcal{M}_{M,N}}(x_1,x_n) \leq E_{\lambda,\mathcal{M}_{M,N}}(x_1,x_2) + E_{\lambda,\mathcal{M}_{M,N}}(x_2,x_3) + \cdots + E_{\lambda,\mathcal{M}_{M,N}}(x_{n-1},x_n).$$

For (ii), we have $\mathcal{M}_{M,N}(x_n, x, \eta) >_{L^*} \mathcal{N}(\lambda) \Leftrightarrow E_{\lambda,\mathcal{M}_{M,N}}(x_n,x) < \eta$ for every $\eta > 0$.

Lemma 2.4: Let $(X, \mathcal{M}_{M,N}, \mathcal{T})$ be a Modified intuitionistic fuzzy metric space. If $\mathcal{M}_{M,N}(x_n, x_{n+1}, t) \geq_{L^*} \mathcal{M}_{M,N}\left(x_0, x_1, \frac{t}{k^n}\right)$ for some $k < 1$ and $n \in \mathbb{N}$ then $\{x_n\}$ is a Cauchy sequence.

Proof. For every $\lambda \epsilon L \backslash \{0_{L^*}, 1_{L^*}\}$ and $x_n \epsilon X$, we have

$E_{\lambda,\mathcal{M}_{M,N}}(x_n, x_{n+1}, t)$

$= inf\{t > 0 : \mathcal{M}_{M,N}(x_n, x_{n+1}, t) >_{L^*} \mathcal{N}(\lambda)\}$

$\leq inf\{t > 0 : \mathcal{M}_{M,N}\left(x_0, x_1, \frac{t}{k^n}\right) >_{L^*} \mathcal{N}(\lambda)\}$

$= inf\{ k^n t : \mathcal{M}_{M,N}(x_0, x_1, t) >_{L^*} \mathcal{N}(\lambda)\}$

$= k^n \; inf\{t > 0 : \mathcal{M}_{M,N}(x_0, x_1, t) >_{L^*} \mathcal{N}(\lambda)\}$

$= k^n E_{\lambda,\mathcal{M}_{M,N}}(x_n, x_{n+1}, t)$

From lemma (2.3), for every $\mu \epsilon L^* \backslash \{0_{L^*}, 1_{L^*}\}$ there exists $\lambda \in L^* \setminus \{0_{L^*}, 1_{L^*}\}$, such that
$E_{\mu,\mathcal{M}_{M,N}}(x_n, x_m) \leq E_{\lambda,\mathcal{M}_{M,N}}(x_n, x_{n+1}) + E_{\lambda,\mathcal{M}_{M,N}}(x_{n+1}, x_{n+2}) + \cdots + E_{\lambda,\mathcal{M}_{M,N}}(x_{m-1}, x_m)$

$\leq k^n E_{\lambda,\mathcal{M}_{M,N}}(x_0, x_1) + k^{n+1} E_{\lambda,\mathcal{M}_{M,N}}(x_0, x_1) + \cdots + k^{m-1} E_{\gamma,\mathcal{M}_{M,N}}(x_0, x_1)$

$= E_{\lambda,\mathcal{M}_{M,N}}(x_0, x_1) \sum_{j=n}^{m-1} k^j \longrightarrow 0.$

Hence sequence $\{x_n\}$ is a Cauchy sequence.

Definition 2.11: Let $(X, \mathcal{M}_{M,N}, \mathcal{T})$ be a Modified intuitionistic fuzzy metric space. $\mathcal{M}$ is said to be continuous on $X \times X \times]0, \infty[$ if

$$\lim_{n\to\infty} \mathcal{M}_{M,N}(x_n, y_n, t_n) = \mathcal{M}_{M,N}(x, y, t),$$

Whenever $\{(x_n, y_n, t_n)\}$ is a sequence in $X \times X \times]0, \infty[$ which converges to a point $(x, y, t) \in X \times X \times]0, \infty[$ that is

$\lim_{n\to\infty} \mathcal{M}_{M,N}(x_n, x, t) = \lim_{n\to\infty} \mathcal{M}_{M,N}(y_n, y, t) = 1_{L^*}$

and $\lim_{n\to\infty} \mathcal{M}_{M,N}(x, y, t_n) = \mathcal{M}_{M,N}(x, y, t).$

Lemma 2.5: Let $(X, \mathcal{M}_{M,N}, \mathcal{T})$ be a Modified intuitionistic fuzzy metric space and for all $x, y \in X$, $t > 0$ and if for a number $k \in (0,1)$, $\mathcal{M}_{M,N}(x, y, kt) \geq_{L^*} \mathcal{M}_{M,N}(x, y, t)$. Then $x = y$

Proof: For every $\lambda \epsilon L \backslash \{0_{L^*}, 1_{L^*}\}$ and $x, y \in X$, we have

$E_{\lambda,\mathcal{M}_{M,N}}(x, y, t)$

$= inf\{t > 0 : \mathcal{M}_{M,N}(x, y, t) >_{L^*} \mathcal{N}(\lambda)\}$

$\leq inf\{t > 0 : \mathcal{M}_{M,N}\left(x, y, \frac{t}{k}\right) >_{L^*} \mathcal{N}(\lambda)\}$

$= inf\{ kt : \mathcal{M}_{M,N}(x,y,t) >_{L^*} \mathcal{N}(\lambda)\}$

$= k \ inf\{t > 0 : \mathcal{M}_{M,N}(x,y,t) >_{L^*} \mathcal{N}(\lambda)\}$

$= kE_{\lambda,\mathcal{M}_{M,N}}(x,y,t)$

Therefore $E_{\lambda,\mathcal{M}_{M,N}}(x,y,t) = 0$.

Hence $x = y$.

Definition 2.12: Let f and g be mappings from a modified intuitionistic fuzzy metric space $(X,\mathcal{M}_{M,N},\mathcal{T})$ into itself. The maps f and g are said to be weakly commuting if $\mathcal{M}_{M,N}(ASz,SAz,t) \geq_{L^*} \mathcal{M}_{M,N}(Az,Sz,t)$ for all $z \in X, t < 0$.

Definition 2.13: A pair of self mappings (f,g) of modified intuitionistic fuzzy metric space $(X,\mathcal{M}_{M,N},\mathcal{T})$ is said to be compatible if $\lim_{n\to\infty} \mathcal{M}_{M,N}(fgx_n, gfx_n, t) = 1_{L^*}$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} = gx_n = z$ for some $z \in X$.

Definition 2.14: Two self mappings f and g are called non compatible if there exist at least one sequence $\{x_n\}$ such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} = gx_n = z$ for some $z \in X$ but either $\lim_{n\to\infty} \mathcal{M}_{M,N}(fgx_n, gfx_n, t) \neq 1_{L^*}$ or the limit does not exist for all $z \in X$.

Definition 2.15: A pair of self mappings (f,g) of modified intuitionistic fuzzy metric space $(X,\mathcal{M}_{M,N},\mathcal{T})$ is said to be weakly compatible if they commute at coincidence point that is if $fu = gu$ for some $u \in X$, then $fgu = gfu$.

Definition 2.16: A pair of self mappings (f,g) of modified intuitionistic fuzzy metric space $(X,\mathcal{M}_{M,N},\mathcal{T})$ is said to be occasionally weakly compatible if the pair (f,g) commutes at least one coincidence point that is there exists at least one point $x \in X$, such that $fx = gx$ and $fgx = gfx$.

Lemma 2.6: Let X be a set, f and g be occasionally weakly compatible self maps on X of a modified intuitionistic fuzzy metric space $(X,\mathcal{M}_{M,N},\mathcal{T})$. If f and g have unique point of coincidence that is $w = fx = gx$ for $x \in X$, then w is the unique common fixed point of f and g.

3 The Main Results

Theorem 3.1: Let $(X,\mathcal{M}_{M,N},\mathcal{T})$ be a complete modified intuitionistic fuzzy metric space and let A,B,S,T,P and Q be self mappings of X. Let the pairs (P,AB) and (Q,ST) be occasionally weakly compatible. If there exist $k \in (0,1)$ such that

$$\mathcal{M}_{M,N}(Px,Qy,kt) \geq_{L^*} min\begin{Bmatrix}\mathcal{M}_{M,N}(ABx,STy,t), \mathcal{M}_{M,N}(Qy,ABx,t),\\ \mathcal{M}_{M,N}(Px,STy,t), \mathcal{M}_{M,N}(ABx,Px,t)\end{Bmatrix} \ldots (1)$$

For all $x, y \in X$ all $t > 0$, then there exist a unique point $w \in X$ such that $Pw = ABw = w$ and a unique point $z \in X$ such that $Qz = STz = z$. Moreover $w = z$, so that there is a unique common fixed point of A, B, S, T, P and Q.

Proof. Let the pairs (P, AB) and (Q, ST) be occasionally weakly compatible, so there are points $x, y \in X$ such that $Px = ABx$ and $Qy = STy$. We claim $Px = Qy$. If not, by inequality (1)

$$\mathcal{M}_{M,N}(Px, Qy, kt) \geq_{L^*} min \begin{Bmatrix} \mathcal{M}_{M,N}(ABx, STy, t), \mathcal{M}_{M,N}(Qy, ABx, t), \\ \mathcal{M}_{M,N}(Px, STy, t), \mathcal{M}_{M,N}(ABx, Px, t) \end{Bmatrix}$$

$$=_{L^*} min \begin{Bmatrix} \mathcal{M}_{M,N}(Px, Qy, t), \mathcal{M}_{M,N}(Qy, Px, t), \\ \mathcal{M}_{M,N}(Px, Qy, t), \mathcal{M}_{M,N}(Px, Px, t) \end{Bmatrix}$$

$$=_{L^*} min \begin{Bmatrix} \mathcal{M}_{M,N}(Px, Qy, t), \mathcal{M}_{M,N}(Qy, Px, t), \\ \mathcal{M}_{M,N}(Px, Qy, t), 1_{L^*} \end{Bmatrix}$$

$$=_{L^*} \mathcal{M}_{M,N}(Px, Qy, t)$$

Therefore $Px = Qy$ that is $Px = ABx = Qy = STy$. Suppose that there is another point z such that $Pz = ABz$ then by inequality (1) we have $Pz = ABz = Qz = STz$. So $Px = Pz$ and $w = Px = ABx$ is the unique point of coincidence of P and AB.

By lemma 2.6, w is the only common fixed point of P and AB. Similarly there is a unique point $z \in X$ such that $z = Qz = STz$.

Assume that $w \neq z$.

We have, by inequality (1)

$$\mathcal{M}_{M,N}(w, z, kt) = \mathcal{M}_{M,N}(Pw, Qz, kt) \geq_{L^*} min \begin{Bmatrix} \mathcal{M}_{M,N}(ABw, STz, t), \mathcal{M}_{M,N}(Qz, ABw, t), \\ \mathcal{M}_{M,N}(Pw, STz, t), \mathcal{M}_{M,N}(ABw, Pw, t) \end{Bmatrix}$$

$$\geq_{L^*} \ min\begin{Bmatrix}\mathcal{M}_{M,N}(w,z,t),\mathcal{M}_{M,N}(z,w,t),\\ \mathcal{M}_{M,N}(w,z,t),\mathcal{M}_{M,N}(w,w,t)\end{Bmatrix}$$

$$=_{L^*} \ min\{\mathcal{M}_{M,N}(w,z,t),\mathcal{M}_{M,N}(z,w,t),\mathcal{M}_{M,N}(w,z,t),1_{L^*}\}$$

$$=_{L^*} \ \mathcal{M}_{M,N}(w,z,t)$$

Therefore we have $z = w$, by lemma 2.6, z is a common fixed point of A, B, S, T, P and Q.

For uniqueness, let u be another common fixed point of A, B, S, T, P and Q. Then

$$\mathcal{M}_{M,N}(z,u,kt) = \mathcal{M}_{M,N}(Pz,Qu,kt) \geq_{L^*} \ min\begin{Bmatrix}\mathcal{M}_{M,N}(ABz,STu,t),\mathcal{M}_{M,N}(Qu,ABz,t),\\ \mathcal{M}_{M,N}(Pz,STu,t),\mathcal{M}_{M,N}(ABz,Pz,t)\end{Bmatrix}$$

$$=_{L^*} \ min\begin{Bmatrix}\mathcal{M}_{M,N}(z,u,t),\mathcal{M}_{M,N}(u,z,t),\\ \mathcal{M}_{M,N}(z,u,t),\mathcal{M}_{M,N}(z,z,t)\end{Bmatrix}$$

$$=_{L^*} \ min\{\mathcal{M}_{M,N}(z,u,t),\mathcal{M}_{M,N}(u,z,t),\mathcal{M}_{M,N}(z,u,t),1_{L^*}\}$$

$$=_{L^*} \ \mathcal{M}_{M,N}(z,u,t)$$

Therefore by lemma 7.6 we have $z = u$.

4 Conclusion

This chapter focuses on establishing common fixed point theorems for mappings that exhibit occasional weak compatibility within the framework of modified intuitionistic fuzzy metric spaces. As a result, the findings presented here offer notable improvements and refinements over several well-established fixed point theorems found in the current literature on metric fixed point theory.

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