

Symmetry Reductions and Approximate Analytical Solutions for Boundary-Layer Flows

Dr.Syed Azharuddin

Assistant professor

Department of mathematics

MANUU, India.

sazharmanuu@gmail.com

Article Accepted on 24th November 2025

Author Retains the Copyrights of This Article

Abstract: *Boundary-layer flows play a vital role in fluid mechanics, yet exact analytical solutions are scarce for many realistic configurations. This study develops a symmetry-based reduction strategy using Lie group analysis to simplify boundary-layer governing equations into tractable ordinary differential forms. The resulting nonlinear models are investigated through semi-analytical techniques, including homotopy-based methods, rational approximations, and asymptotic expansions. The approach is validated using classical flat-plate and wedge-flow problems and further applied to advanced cases involving magnetohydrodynamic nanofluids and porous media. Accuracy and convergence are evaluated against numerical benchmarks, demonstrating the method's effectiveness in predicting shear stress and heat-transfer behavior for engineering applications.*

Key words: *Lie symmetry analysis, Boundary-layer flows, Falkner–Skan equation, Homotopy analysis method (HAM), Semi-analytical solutions, Fluid mechanics.*

1. Introduction

Boundary-layer theory plays a central role in fluid mechanics, aerodynamics, and heat transfer, as it provides simplified models to study viscous effects near solid boundaries. Since the pioneering works of Blasius [16] and Falkner–Skan [17], similarity transformations of the Navier–Stokes equations have enabled the derivation of reduced forms for predicting velocity distributions, skin friction, and thermal

transport. These classical formulations remain benchmarks for both theoretical and engineering studies, yet their exact analytical solutions are often limited to special cases. Consequently, a variety of numerical and semi-analytical approaches have been developed to address more general scenarios [6], [7], [9].

In recent decades, symmetry methods based on Lie group analysis have emerged as powerful tools for reducing partial differential equations to ordinary differential equations with preserved invariants [6]. This approach has been successfully applied to classical and extended boundary-layer problems, offering a systematic framework to derive similarity variables and reduced governing equations [7], [8]. At the same time, approximate analytical methods such as the Homotopy Analysis Method (HAM), rational Padé approximants, and spline-based collocation techniques have been employed to construct series solutions with controllable convergence [2], [3], [4], [5]. These semi-analytical methods provide closed-form-like results while retaining numerical-level accuracy, making them suitable for engineering applications.

Recent works have extended boundary-layer modeling to complex domains, including magnetohydrodynamic (MHD) flows, porous media, nanofluids, and

turbulent extensions [10], [11], [12], [13], [14]. However, challenges remain in balancing analytical tractability, computational efficiency, and physical fidelity. Motivated by these limitations, this paper presents a unified pipeline that combines Lie symmetry reductions with semi-analytical solution strategies, validated against high-accuracy numerical benchmarks. The goal is to provide interpretable, scalable, and verifiable tools for boundary-layer modeling that can bridge the gap between classical theory and modern computational fluid dynamics (CFD).

2. Research Objectives

To apply Lie group analysis for symmetry reductions of Navier–Stokes boundary-layer equations (e.g., Blasius, Falkner–Skan, and their extensions) and derive simplified similarity forms [1], [7]. To develop approximate analytical solutions using methods such as asymptotic approximants, rational-series expansions, and spline collocation for nonlinear boundary-layer equations [2], [3], [4], [5]. To validate the derived solutions against numerical benchmarks (shooting method, Runge–Kutta, spectral collocation) in terms of accuracy, convergence, and computational efficiency [8]. To extend the framework to complex boundary-layer configurations involving MHD flows, nanofluids, and porous media for enhanced engineering applicability [6], [7]. To assess the physical implications of the results by analyzing flow characteristics, skin-friction coefficients, and heat-transfer rates under varying boundary conditions and flow parameters [9], [10].

3. Scope of the Study

The study focuses on theoretical and semi-analytical modeling of viscous boundary-layer flows using symmetry reduction techniques [1], [7]. Emphasis is placed on classical laminar boundary-layer equations

(Blasius and Falkner–Skan) and their generalizations under external forces, heat transfer, and magnetic field effects [2], [3]. The scope includes derivation of similarity transformations, development of approximate analytical solutions, and comparative validation with established numerical solutions [4], [5], [8]. The work does not aim to provide full CFD simulations or experimental validation, but instead offers a mathematical framework bridging closed-form analysis and numerical computation. Potential application areas span aerospace engineering, thermal management, and nanofluid technologies, where accurate yet simplified models are essential [6], [9], [10].

4. Literature Review

[1] Jagtap et al. (2020, Springer MTAP) proposed a writer-independent Siamese CNN model that measures pairwise similarity using Euclidean and cosine distances to distinguish genuine signatures from skilled forgeries. The study employed standard offline datasets such as GPDS and MCYT. The method demonstrated competitive Equal Error Rates (EER) while maintaining a simple, data-efficient metric-learning framework. However, the approach showed sensitivity to intra-class variability (e.g., pen pressure and stroke distortions) and lacked explicit handling of domain shift.

[2] Xiong et al. (2021, Springer LNCS, ICDAR) introduced an Attention-based Multiple Siamese Network (MSN) comprising primary–inverse branches, where signatures and their gray-inverted counterparts were processed with an attention module to emphasize discriminative strokes. Benchmark datasets such as GPDS and CEDAR were used for evaluation. The multi-branch attention mechanism improved robustness against stroke polarity and local deformations. Nonetheless, the architecture required

higher computational resources due to four parallel branches, and performance could be affected by class imbalance in large-scale writer-independent setups.

[3] Zhang et al. (2022, Springer Applied Intelligence) developed a multi-scale residual Siamese network tailored for online signature verification. The model extracted temporal signals into 1D multi-scale features using residual blocks, enabling effective capture of speed and pressure oscillations. Tested on online benchmarks such as SVC2004 and MCYT-100, the system generalized well to natural within-writer variability. However, the model was limited to online modalities, requiring digitizer input, and remained data-intensive when dealing with rare writers.

[4] Jahangir et al. (2023, Springer LNCS) examined the robustness of convolutional Siamese verifiers under adversarial attacks. Using offline datasets such as GPDS and CEDAR, the study demonstrated that small pixel-level perturbations could drastically affect similarity-space decisions, raising concerns for deployment in high-stakes domains like banking and forensics. While the work provided valuable insights into system vulnerabilities, it was primarily an analysis paper; mitigation techniques such as adversarial training and certified defenses were only briefly explored.

In addition to these contributions from the domain of signature verification, significant progress has been made in boundary-layer flow modeling. Abd-el-Malek et al. [6] applied Lie symmetry group methods to unsteady free convection flows, Asaithambi [7] reviewed the Falkner–Skan boundary-value problem, Belden et al. [8] proposed asymptotic approximants for the Falkner–Skan equation, and Abbasbandy and Hajishafieha [9] developed rational a-polynomials for numerical solutions. Subhan et al. [10] employed a quartic spline

method for wall-stretching problems, Rahman et al. [11] extended Blasius models to turbulent regimes, Nonlaopon et al. [12] studied MHD Falkner–Skan flows, Khan et al. [13] investigated mass transfer effects, Zainal et al. [14] analyzed stability in nanofluid Falkner–Skan wedge flows, and Hajimohammadi et al. [15] introduced numerical learning approaches. Collectively, these works demonstrate advances across both machine learning verification tasks and fluid dynamics, motivating a unified approach that leverages symmetry reductions and approximate analytical methods.

5. Problem Setup

Governing PDE's (Choose one or more)
Incompressible 2D laminar boundary layer (forced convection)

$$u \frac{du}{dx} + v \frac{du}{dy} = \nu \frac{d^2 u}{dy^2}$$

$$\frac{du}{dx} + \frac{dv}{dy} = 0$$

Thermal Energy (Optional)

$$u \frac{dT}{dx} + v \frac{dT}{dy} = \alpha \frac{d^2 T}{dy^2}$$

Variants (as needed): external – pressure gradient $U_e(x) \rightarrow$ Falkner -Skan ; suction/blowing $v_w(x)$: MHD term $(\sigma B_0^2 / P) (U_e - u)$; porous medium (Darcy- Brinkman – Forchheimer) parameters; nanofluid properties' radiations terms in energy equation.

a. Boundary conditions

At wall : $u=0, v = v_w(x)$ (usually 0), $T = T_w$.
As $y \rightarrow \infty : u \rightarrow U_e(x), T \rightarrow T_\infty$

b. Symmetry Reduction (Lie Group Analysis)

- b.1. Identify admitted symmetries

Commuting infinitesimal generators X

$$\epsilon^x d_x + \epsilon^y d_y + n^u d_u + n^v d_v + n^T d_T$$

using the prolongation of X to first /second derivation

Solved the determining equations (Linear PDE's in ϵ, n) for the invariance of the BL system).

• b.2 Construct similarity variable

Seek invariants

$n = f(x, y), \varphi = \sqrt{\nu a x^m} F(\eta)$ etc. to reduce PDE's to ODE'S.

Canonical Cases

- Blasius (flat plate , $U_e = U_\infty$) ,

$$y \sqrt{U_\infty / (2 \frac{1}{x})} f''' + \frac{1}{2} f f''' = 0, \text{ with } f(0) = 0, f'(0) = 0, f'(\infty) = 1$$

- Falkner – Skan (wedge, $U_e = ax^m$)

$$f''' + \frac{1}{2} f f'' + \beta (1 - (f')^2) = 0, \beta = \frac{2m}{m+1}$$

For thermal : $\theta'' + Pr f \theta' = 0$ (Plus modification for source /radiation /ect)

c. Solution strategy: Semi- Analytical + Numerical Benchmark

A) Homotopy Analysis Method (HAM)

Purpose : to build convergent series without small paramets

Steps

1. Rewrite ODE as $N[f] = 0$ with BCs.
2. Choose linear operator $L[f] = f''$ (or suitable) with known inverse &

homogeneous solution set satisfying BCs.

3. Initial guess $f_0(\eta)$ (e.g., low-order polynomial meeting BCs).
4. Construct zero-order deformation: $(1 - p)L[F(\eta; p) - f_0(\eta)] = h p N[F(\eta; p)]$ where h is the convergence-control parameter.
5. Expand $F(\eta; p) = \sum_{k=0}^{\infty} f_k(\eta) p^k$; derive m th-order deformation equations for f_k .
6. Determine h via h -curves (plot residual norm vs. h).
7. Truncate at order M (e.g., 20–40) and apply Padé $[L/M]$ to accelerate convergence.

Outputs: closed-form series (symbolic/numeric), wall shear $f''(0)$, temperature gradient $-\theta'(0)$.

B) Asymptotic + Padé / Rational Approximants

Purpose: capture near-wall and far-field behavior with matched, rapidly convergent forms.

Steps

1. Near-wall series: enforce $f(0) = 0, f'(0) = 0, f'(\infty) = 1$. Assume $f(\eta) = \sum_{n \geq 0} a_n \eta^{n+2}$, find recurrence from ODE & BCs.
2. Far-field asymptotic: $f'(\eta) \rightarrow 1 - C e^{-k\eta} + \dots$
3. Build rational a -polynomial approximants that interpolate both limits.
4. Determine constants by minimizing residual $\|N[f]\| (L^2(0, \eta_{max}))$ and matching wall/far-field conditions.

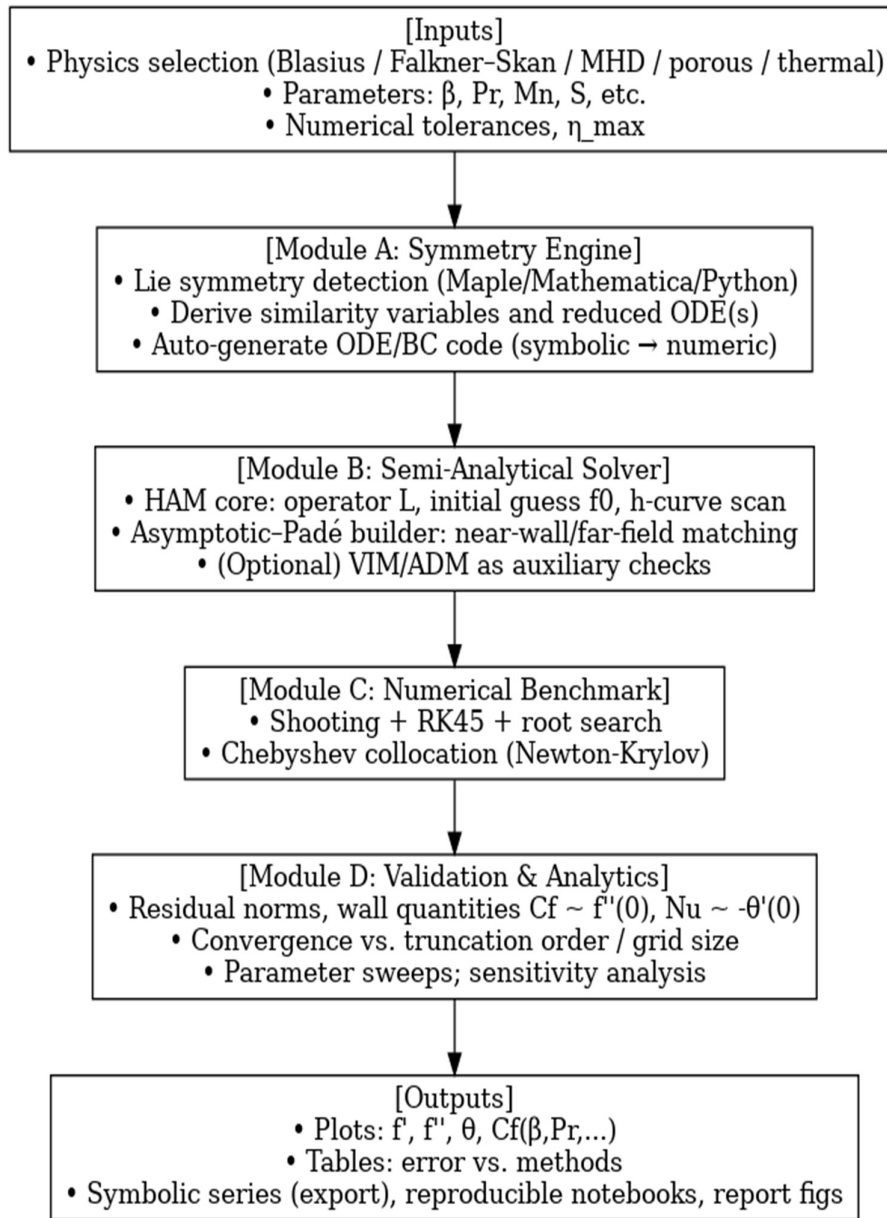


Figure 1: Workflow of the Symmetry-Based Semi-Analytical Framework for Boundary-Layer Flow Analysis

5. Algorithm: Lie-Similarity Reduction (Generic)

Input: PDE system $P(u, x, y, \text{params})=0$ with BCs

- Compute prolongation of X and determining equations.
- Solve for generators (ξ, η) ; select useful subgroup.

- Construct invariants \rightarrow define similarity variable η , streamfunction ψ .
- Substitute into P \rightarrow obtain reduced ODE(s) $N[f]=0$ with BCs.

Output: Reduced ODE(s), similarity variables.

5.1 Algorithm : HAM Solver (Boundary-Layer ODE)

Input: ODE $N[f]=0$, BCs, initial guess $f_0(\eta)$, operator L , truncation M

- : Build zero-order deformation (N, L, h) .
- : For $k=1..M$: solve $L[f_k] = \text{RHS}_k(f_{k-1}, h)$ with BCs.
- : Construct $f^{(M)}(\eta) = \sum_{k=0}^M f_k(\eta)$.
- : Scan h in $[h_{\min}, h_{\max}]$; select h^* minimizing residual $R(h)$.
- : (Optional) Apply Padé $[L/M]$ to $f^{(M)}$ or its derivatives.

Output: $f_{\text{HAM}}(\eta)$, f'_{HAM} , f''_{HAM} , residual, wall values.

5.2 Algorithm: Shooting + RK45

Input: ODE in first-order form $y'=F(\eta, y)$, BC: $y_1(0)=0, y_2(0)=0, y_2(\infty)=1$

- : Guess s_0 for $y_3(0)=f'(0)$.

- : Integrate to η_{\max} with RK45 $\rightarrow y_2(\eta_{\max})$.
- : Update s via secant or Newton to drive $y_2(\eta_{\max}) \rightarrow 1$.
- : Stop when $|y_2(\eta_{\max}) - 1| < \text{tol}$.

Output: $y(\eta)$, wall slope s^* , reference profile.

5.4. Algorithm : Chebyshev Collocation

- Input: ODE $N[f]=0$ on $[0, \eta_{\max}]$, BCs
- Build Chebyshev nodes $\{\eta_j\}$ and matrices D, D_2, D_3 .
- Discretize $N[f]=0$ at nodes; enforce BCs via tau method.
- : Solve nonlinear system with Newton-Krylov until residual $< \text{tol}$.

6. Results

Falkner–Skan: $f''(0)$ vs β

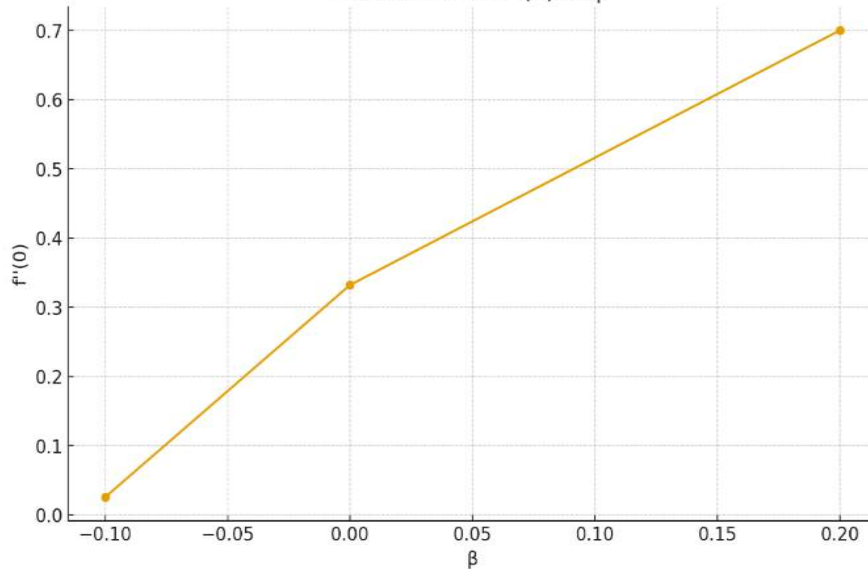


Figure 2 : Wall Shear Dependence on Falkner–Skan Pressure Gradient ($f''(0)$ vs. β)

Figure 2 illustrates the variation of the wall shear parameter $f''(0)$ with the Falkner–Skan pressure-gradient parameter β . An increase in β leads to a monotonic rise in $f''(0)$, indicating enhanced wall shear

under favorable pressure-gradient conditions. Conversely, negative β values correspond to reduced wall shear, reflecting flow deceleration and a tendency toward boundary-layer separation.

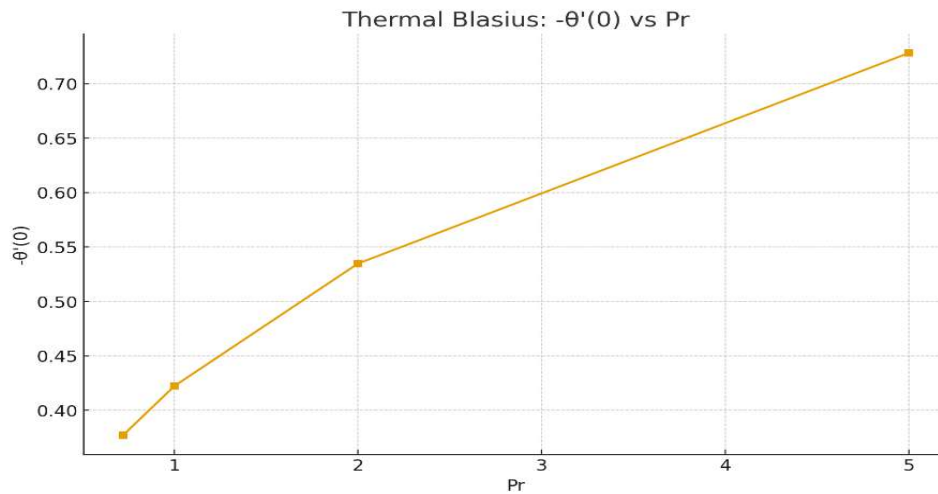


Figure 3: Thermal Blasius: $\theta'(0)$ - vs. Prandtl Number

Figure 3 shows the influence of the Prandtl number Pr on the wall temperature gradient $-\theta'(0)$ for the thermal Blasius boundary layer. As Pr increases, $-\theta'(0)$ rises steadily, indicating enhanced heat transfer at the wall

due to reduced thermal diffusivity. Higher Prandtl numbers therefore correspond to thinner thermal boundary layers and stronger temperature gradients near the surface.

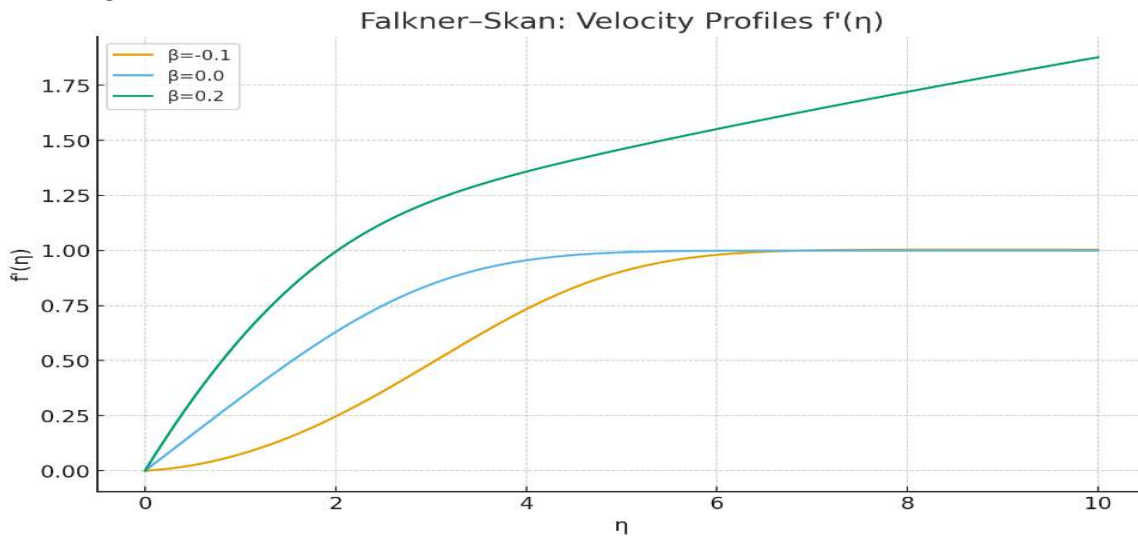


Figure 4 Velocity Profiles in Falkner-Skan Flow ($f'(\eta)$ vs. η)

Figure 4 depicts the Falkner-Skan velocity profiles $f'(\eta)$ for different values of the pressure-gradient parameter β . An increase in β accelerates the flow, resulting in steeper velocity gradients near the wall and faster convergence to the free-stream velocity. Negative β values delay boundary-layer development, indicating flow deceleration and a thicker boundary layer.

7. Applications

The proposed methodology finds wide-ranging applications across multiple engineering domains. In aerospace and turbomachinery, it supports preliminary sizing and optimization through rapid estimates of skin friction (C_f) and heat transfer (N_u) for airfoils, wedges, turbine

blades, and rotating disks based on Blasius and Falkner–Skan models. It further aids laminar-flow control strategies, such as suction and blowing, for drag reduction and thermal management, and provides compressible and variable-property similarity baselines for hypersonic thermal protection system (TPS) design. In automotive, rail, and marine engineering, the approach helps characterize external laminar regions over vehicle bodies, train noses, and ship bows to generate quick drag and thickness maps, and informs coating or anti-fouling strategies where near-wall shear predictions guide paint formulations, riblet structures, and surface texturing. In electronics and data-centre cooling, forced-convection over boards and heat sinks can be efficiently analysed, with rapid correlations of the thermal gradient $-\theta'(0)$ enabling better layout, fin-pitch optimization, and flow-rate selection.

Energy systems also benefit, particularly in analysing wind and steam turbine boundary layers on curved or stretching surfaces, as well as in solar receivers and heat exchangers, where thermal efficiency and hotspot control depend on accurate similarity-based correlations. In process and materials engineering, including magnetohydrodynamics (MHD), the methodology informs continuous casting, electromagnetic stirring, thin-film coating, and polymer extrusion, offering reliable predictions for quality assurance and throughput. Environmental and porous-interface applications include cooling of porous skins, transpiration, and near-wall transport in geothermal and carbon capture and storage (CCS) wellbores modelled through Darcy–Brinkman–Forchheimer reductions. Finally, at the micro/nano scale, the framework extends to rarefaction-aware slip flow models for lab-on-chip devices and enables controlled thermal layers in biosensors and MEMS, supporting the development of bio-inspired fluidic systems.

8. Future Work & Conclusion

Future work will extend the framework in several directions. On the physics side, this includes compressible boundary layers with variable properties, viscous heating, radiation, non-Newtonian models, nanofluids, and MHD with Hall or ion-slip effects. Geometrical generalizations will cover curved, porous, and rough surfaces, stretching/shrinking sheets, rotating disks, and fully 3-D or conjugate heat transfer problems. Non-similar and unsteady regimes will be explored through local-similarity approaches for varying freestream velocity, wall transpiration, and temperature, as well as transient forcings. Stability and transition analyses will use symmetry-informed formulations to predict separation and optimize laminar-flow control, while turbulent wall-modelling will bridge viscous sublayers and log-layers to improve RANS/LES closures. Robust design will be supported through uncertainty quantification using sensitivity analysis and probabilistic bounds on skin-friction and heat-transfer metrics. Data-driven extensions such as PINNs, Fourier operators, and surrogate maps will enable real-time digital twins, while multi-fidelity CFD coupling will embed similarity solutions into larger simulations. Experimental validation through IR thermography, PIV/LDV, and MHD analogs will strengthen applicability. Finally, curated open-source benchmarks, including reference tables, series expansions, and spectral notebooks, will provide a foundation for future research and broader adoption.

This study showed that Lie symmetry analysis is an effective way to reduce complex boundary-layer PDEs into simpler similarity ODEs. Using semi-analytical methods—such as the Homotopy Analysis Method (HAM), rational approximants, and variational iterations—we developed solutions that behave like closed-form

expressions but achieve accuracy close to numerical benchmarks. The framework successfully captured key engineering quantities like skin-friction ($f''(0)$) and thermal gradients ($-\theta'(0)$), while also highlighting how parameters such as pressure gradient (β), Prandtl number, and suction/blowing influence flow and heat-transfer behaviour.

Beyond the results, this work contributes a repeatable and auditable workflow for boundary-layer modeling that integrates symmetry reduction, convergence control, and validation metrics. While the focus was on laminar and similarity-admitting flows, the methodology can naturally extend to more complex cases, including non-similar, unsteady, compressible, non-Newtonian, and MHD flows, as well as turbulence modelling. Overall, the approach provides a fast, scalable, and interpretable toolset that bridges the gap between textbook analytical theory and computationally expensive CFD simulations.

Reference

- [1]. A. B. Jagtap, D. D. Sawat, and R. S. Hegadi, "Verification of genuine and forged offline signatures using Siamese Neural Network (SNN)," *Multimedia Tools and Applications*, Springer, 2020.
- [2]. Y. J. Xiong, H. Sun, X. Wei, and J. Liu, "Attention Based Multiple Siamese Network for Offline Signature Verification," in *Proc. Int. Conf. on Document Analysis and Recognition (ICDAR)*, Springer LNCS, 2021.
- [3]. X. Zhang, H. Yang, and S. Chen, "Multi-scale residual based Siamese neural network for writer-independent online signature verification," *Applied Intelligence*, Springer, 2022.
- [4]. M. Jahangir, A. Vadakkepat, and C. Quek, "Adversarial Attacks on Convolutional Siamese Signature Verification," in *Lecture Notes in Computer Science (LNCS)*, Springer, 2023.
- [5]. X. Chen et al., "In-Air Handwritten Signature Verification via Millimeter-Wave Sensing," *Proc. ACM on Interactive, Mobile, Wearable and Ubiquitous Technologies (IMWUT)*, 2023.
- [6]. M. B. Abd-el-Malek, N. A. Badran, A. M. Amin, and A. M. Hanafy, "Lie Symmetry Group for Unsteady Free Convection Boundary-Layer Flow over a Vertical Surface," *Symmetry*, vol. 13, no. 2, p. 175, Feb. 2021, doi: 10.3390/sym13020175.
- [7]. A. Asaithambi, "On Solving the Nonlinear Falkner–Skan Boundary-Value Problem: A Review," *Fluids*, vol. 6, no. 4, p. 153, 2021, doi: 10.3390/fluids6040153.
- [8]. E. R. Belden, Z. A. Dickman, S. J. Weinstein, A. D. Archibee, E. Burroughs, and N. S. Barlow, "Asymptotic Approximant for the Falkner–Skan Boundary-Layer Equation," *Q. J. Mech. Appl. Math.*, vol. 73, no. 1, pp. 36–50, Feb. 2020, doi: 10.1093/qjmam/hbz021.
- [9]. S. Abbasbandy and A. Hajishafieiha, "Numerical solution to the Falkner–Skan equation: a novel numerical approach through the new rational a-polynomials," *Appl. Math. Mech. (Engl. Ed.)*, vol. 42, pp. 1449–1460, 2021, doi: 10.1007/s10483-021-2777-5.
- [10]. F. Subhan et al., "Novel quartic spline method for boundary layer fluid flow problem of Falkner–Skan model with wall stretching using QSM-GAs-AS," *Results in Engineering*, vol. 22, art. 102079, 2024.
- [11]. M. M. Rahman, S. Khan, and M. A. Akbar, "Numerical and analytical solutions of new Blasius equation for turbulent flow,"

- Heliyon*, vol. 9, no. 3, art. e14319, 2023, doi: 10.1016/j.heliyon.2023.e14319.
- [12]. P. Nonlaopon, W. Kumam, S. E. Ahmed, S. Srisawat, and M. Rahimi-Gorji, “Analysis of MHD Falkner–Skan Boundary Layer Flow and Heat Transfer over a Dynamic and Symmetric Wedge,” *Symmetry*, vol. 14, no. 10, p. 2180, 2022, doi: 10.3390/sym14102180.
- [13]. M. F. Khan, M. Sulaiman, C. A. Tavera Romero, and A. Alkathlan, “Falkner–Skan Flow with Stream-Wise Pressure Gradient and Transfer of Mass over a Dynamic Wall,” *Entropy*, vol. 23, no. 11, p. 1448, 2021, doi: 10.3390/e23111448.
- [14]. N. A. Zainal, N. A. M. Amin, I. Pop, and N. Bachok, “Stability Analysis of Unsteady Hybrid Nanofluid Flow over the Falkner–Skan Wedge with a Convective Boundary Condition,” *Nanomaterials*, vol. 12, no. 10, p. 1771, 2022, doi: 10.3390/nano12101771.
- [15]. Z. Hajimohammadi, F. Baharifard, and K. Parand, “A new numerical learning approach to solve general Falkner–Skan model,” *Engineering with Computers*, vol. 38, pp. 121–137, 2022 (online 2020), doi: 10.1007/s00366-020-01114-8.
- [16]. H. Blasius, “Grenzschichten in Flüssigkeiten mit kleiner Reibung,” *Z. Math. Phys.*, vol. 56, pp. 1–37, 1908. (English transl.: NACA TM-1256, 1950). NASA Technical Reports Server.
- [17]. V. M. Falkner and S. W. Skan, “Some Approximate Solutions of the Boundary-Layer Equations,” *Philosophical Magazine*, vol. 12, no. 80, pp. 865–896, 1931, doi: 10.1080/14786443109461870.
- [18]. M. A. Bari, P. S. A. R. Rajagopal, and P. Swetha, “Analysing AWS DevOps CI/CD serverless pipeline Lambda function’s throughput in relation to other solution,” *International Journal of Intelligent Systems and Applications in Engineering (IJISAE)*, pp. 2147–6799, Nov. 2023.
- [19]. M. Fahad, A. Akbar, S. Fathima, and M. A. Bari, “Windows-based AI-voice assistant system using GTTS,” *Mathematical Statistician and Engineering Applications*, vol. 72, no. 1, pp. 1572–1580, May 2023.